can be noted that $|S_{11}|$ (or $|S_{22}|$) has the same upper bound as that deriving from condition (1), but this limit corresponds only to the value of $|S_{22}|$ (or $|S_{11}|$) given by

$$|S_{22}| = (|S_{12}/S_{21}|)(1 - |S_{21}|^2)^{1/2}.$$

If three moduli are given, e.g., $|S_{22}|$, $|S_{12}|$, and $|S_{21}|$, the bounds imposed by passivity on the last, $|S_{11}|$, are the following:

$$\frac{|S_{22}||S_{12}||S_{21}| - [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2)]^{1/2}}{1 - |S_{22}|^2} < |S_{11}| < \frac{|S_{22}||S_{12}||S_{21}| + [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2)]^{1/2}}{1 - |S_{22}|^2}. \quad (7)$$

The lower bound in condition (7) is greater than zero only when

$$|S_{22}| > [(1 - |S_{12}|^2)(1 - |S_{21}|^2)]^{1/2} \quad (8)$$

as also clearly results from Fig. 1.

C. Passivity Conditions for Any Value of the Arguments

When the values of the moduli are such that condition $L > 1$ is satisfied the arguments of the S parameters can assume every value. In Fig. 1 the allowed region for $|S_{11}|$ and $|S_{22}|$ is now that bounded by the axes and the curve C_0.

In particular, when $|S_{22}|$, $|S_{12}|$, and $|S_{21}|$ are given, in order that the passivity conditions for any value of the arguments hold, the last modulus, $|S_{11}|$, must satisfy the condition

$$|S_{11}| < \frac{-|S_{22}||S_{12}||S_{21}| + [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2)]^{1/2}}{1 - |S_{22}|^2}. \quad (9)$$

IV. RECIPROCAL NETWORKS

When the two-port network is reciprocal ($S_{12} = S_{21}$), the minimum realizability condition (7) becomes

$$\frac{|S_{12}|^2}{1 - |S_{22}|} - 1 < |S_{11}| < 1 - \frac{|S_{12}|^2}{1 + |S_{22}|}. \quad (10)$$

The lower bound in condition (9) is positive only if

$$|S_{22}| > 1 - |S_{12}|^2. \quad (11)$$

Furthermore, the passivity conditions for any value of the arguments (9) reduces to

$$0 \leq |S_{11}| < 1 - \frac{|S_{12}|^2}{1 - |S_{22}|}. \quad (12)$$

The same relations hold for $|S_{22}|$, exchanging $|S_{11}|$ with $|S_{22}|$.

As regards the limits imposed on the modulus of the transmission coefficient S_{12}, when $|S_{11}|$ and $|S_{22}|$ are given they are easily found from condition (2).

In the case of the minimum passivity conditions, one obtains [6]

$$|S_{12}| < [(1 + |S_{11}|)(1 - |S_{22}|)]^{1/2} \quad (13)$$

when $|S_{11}| \leq |S_{22}|$, and

$$|S_{12}| < [(1 - |S_{11}|)(1 + |S_{22}|)]^{1/2} \quad (14)$$

when $|S_{11}| \geq |S_{22}|$.

If the passivity conditions for any value of the arguments are to be verified, it must be

$$|S_{12}| < [(1 - |S_{11}|)(1 - |S_{22}|)]^{1/2}. \quad (15)$$

The latter conditions are also directly deducible from Fig. 1, exchanging transmission with reflection coefficients and observing that, in the reciprocal case, the allowed values of $|S_{12}| = |S_{21}|$ are lying on the bisecting line.

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Multiconductor Transmission Lines and the Green's Matrix

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***Abstract*—It is shown that the study of arbitrarily terminated multiconductor transmission lines which may in general be lossy and subjected to excitation applied at an arbitrary point along the lines, may be effectively performed with the aid of the appropriate Green's matrix. The procedure is illustrated using the Chipman method of impedance measurement as well as coupled microstrip lines.**

I. INTRODUCTION

There is a considerable body of literature dealing with multiconductor transmission lines [1]–[4] which are encountered in such diverse contexts as microstrip directional couplers, overmoded waveguides, shielded pair instrumentation cables, etc., to mention but a few applications.

While known procedures are useful in dealing with various specialized cases, it will be shown that the use of the Green's matrix technique makes it possible to deal with a very wide range of situations, including excitation by voltage and current sources applied at points not necessarily coinciding with the end terminals.

In what follows we shall consider a multiconductor transmission line comprising n distinct conductors (which may be lossy) in addition to the ground path. We find that [2] in the sinusoidal steady state

$$\begin{bmatrix} V' \\ I' \end{bmatrix} = \begin{bmatrix} 0 & -Z \\ -Y & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} + \begin{bmatrix} E_S \\ I_S \end{bmatrix} \quad (1)$$

where V, I, V', I', E_S, and I_S are n-dimensional column vectors while Z and Y are $n \times n$ square matrices. Furthermore, V, I, Z, Y, E_S, I_S, and x have the usual meaning of voltage, current, impedance per unit length, admittance per unit length, applied voltage per unit length, applied current per unit length and distance, respectively, while a prime denotes differentiation with respect to x.

II. THE MULTICONDUCTOR LINE GREEN'S MATRIX

With reference to Cole [5], a system of $2n$ differential equations

$$u' = Au + f(x) \quad (2)$$

subject to two point boundary conditions $W^a u(a) + W^b u(b) = 0$ has a solution of the form

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$$u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi \tag{3}$$

where

$$G(x,\xi) = \Phi(x) D^{-1} W^a \Phi(a) \Phi^{-1}(\xi), \qquad \xi \leq x \tag{4a}$$

$$G(x,\xi) = -\Phi(x) D^{-1} W^b \Phi(b) \Phi^{-1}(\xi), \qquad \xi > x. \tag{4b}$$

A, W^a, W^b, D^{-1}, Φ, $G(x,\xi)$ are $2n \times 2n$ square matrices, $u(x)$ and $f(x)$ are $2n$-dimensional vectors. Furthermore, Φ represents an arbitrary fundamental matrix of the homogeneous counterpart of (2), $G(x,\xi)$ denotes the Green's matrix while D is defined by $D = W^a\Phi(a) + W^b\Phi(b)$ and it is assumed that D is nonsingular.

For reference, (4) can be readily derived when it is noted that the solution of (2) has a general form $u = y + \Phi c$ where y is a particular vector solution and c is a yet unspecified matrix independent from x, which can be determined with the aid of the boundary conditions. With due regard to the foregoing boundary conditions and the definition of D, it follows that $c = -D^{-1}[W^a y(a) + W^b y(b)]$. It is easily shown that [5]

$$y(x) = \int_a^x \Phi(x) \Phi^{-1}(\xi) f(\xi)\, d\xi$$

and (4a) follows at once; (4b) is derived with the aid of a few further algebraic transformations.

Applying (4a) and (4b) to a multiconductor transmission line (Fig. 1) of length L, we note [1],[2] that a fundamental matrix [as can be readily verified by evaluating the state transition matrix $\exp(Ax)$] satisfying (1) has the form

$$\Phi(x) = \begin{bmatrix} \cosh \Gamma x & -\sinh \Gamma x (\Gamma^{-1} Z) \\ -\sinh \Gamma^T x [(\Gamma^T)^{-1} Y] & \cosh \Gamma^T x \end{bmatrix} \tag{5}$$

where $\Gamma = (ZY)^{1/2}$ and $\Gamma^T = (YZ)^{1/2}$. However, there are computational advantages in using a different fundamental matrix (not the transition matrix), viz.,

$$\Phi(x) = \begin{bmatrix} \exp(-\Gamma x) & \exp(\Gamma x) \\ \exp(-\Gamma^T x) Y_0 & -\exp(\Gamma^T x) Y_0 \end{bmatrix} \tag{6}$$

where $Y_0 = Z^{-1}\Gamma$. Equation (6) follows from (5) when it is noted that $\Phi(x)$ in (4) is determinate only within a nonsingular constant matrix.

Although a wide range of boundary conditions can be handled, for the purpose of illustrating the procedure we shall assume that the multiconductor lines are terminated in impedances Z_a at $x = 0$ and Z_b at $x = L$ [$x = 0$, $x = L$ corresponding to $x = a$ and $x = b$, respectively, in (3)] where Z_a and Z_b are $n \times n$ diagonal matrices. Since the boundary conditions are $V(0) + Z_a I(0) = V(L) - Z_b I(L) = 0$, it follows that

$$W^a = \begin{bmatrix} U & Z_a \\ 0 & 0 \end{bmatrix} \qquad W^b = \begin{bmatrix} 0 & 0 \\ U & -Z_b \end{bmatrix} \tag{7}$$

and with reference to (6)

$$D = \begin{bmatrix} U + Z_a Y_0 & U - Z_a Y_0 \\ \exp(-\Gamma L) - Z_b \exp(-\Gamma^T L) Y_0 & \exp(\Gamma L) + Z_b \exp(\Gamma^T L) Y_0 \end{bmatrix} \tag{8}$$

where U represents a unity matrix. Furthermore, with reference to (1) and (3)

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \int_0^L \begin{bmatrix} G_{11}(x,\xi) & G_{12}(x,\xi) \\ G_{21}(x,\xi) & G_{22}(x,\xi) \end{bmatrix} \begin{bmatrix} E_S(\xi) \\ I_S(\xi) \end{bmatrix} d\xi \tag{9}$$

after suitably partitioning the Green's matrix $G(x,\xi)$; evidently, (9) holds for both point sources and distributed sources.

In particular, in the absence of an applied current source I_S, and corresponding to excitation due to an induced voltage source E_S only, we find that for values of x past the point at which the source is located, i.e., $\xi \leq x$

$$G_{11} = -MRP \qquad G_{12} = MRQ \qquad G_{21} = -MSP \qquad G_{22} = MSQ \tag{10}$$

where

$$M = -1/\{\det[Y_0(\exp[(\Gamma^T - \Gamma)\xi] + \exp[(\Gamma - \Gamma^T)\xi])] \cdot \det[\exp(\Gamma L)(U + Z_a Y_0) + \exp(\Gamma^T L)(Z_b Y_0 + Z_a Z_b Y_0^2) + \exp(-\Gamma L)(Z_a Y_0 - U) + \exp(-\Gamma^T L)(Z_b Y_0 - Z_a Z_b Y_0^2)]\} \tag{11}$$

$$P = [(U + Z_a Y_0)\exp(\Gamma^T \xi) + (U - Z_a Y_0)\exp(-\Gamma^T \xi)] Y_0 \tag{12}$$

$$Q = (U - Z_a Y_0)\exp(-\Gamma\xi) - (U + Z_a Y_0)\exp(\Gamma\xi) \tag{13}$$

$$R = \exp(-\Gamma x)[\exp(\Gamma L) + Z_b \exp(\Gamma^T L) Y_0] + \exp(\Gamma x)[Z_b \exp(-\Gamma^T L) Y_0 - \exp(-\Gamma L)] \tag{14}$$

$$S = \exp(-\Gamma^T x) Y_0 [\exp(\Gamma L) + Z_b \exp(\Gamma^T L) Y_0] - \exp(\Gamma^T x) Y_0 [Z_b \exp(-\Gamma^T L) Y_0 - \exp(-\Gamma L)]. \tag{15}$$

The numerical computation of matrix functions such as Γ, $\exp(\Gamma x)$, etc., may be effectively performed with the aid of Sylvester's theorem [1], [6] which holds when all eigenvalues are distinct and also when some or all are equal [6]. If the system consists of lossless conductors in a homogeneous dielectric, all eigenvalues are equal and hence $\Gamma = \Gamma^T$; for systems comprising, for example, inhomogeneous dielectrics or lossy conductors, the eigenvalues are in general distinct.

As mentioned earlier, the Green's matrix technique is also applicable to other boundary conditions than those given by (7) such as those due to lumped element coupling between the lines at one or both boundaries (Fig. 2).

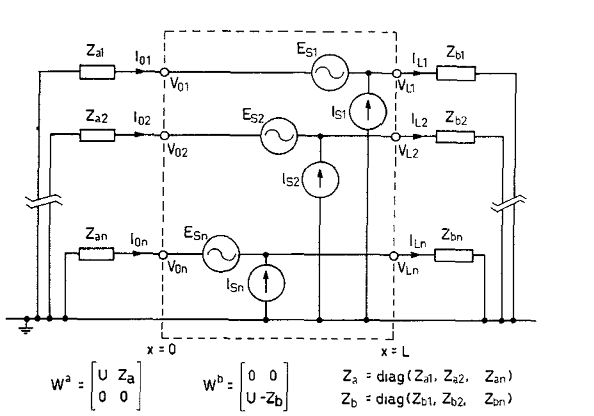

Fig. 1. Multiconductor transmission line terminated at both ends.

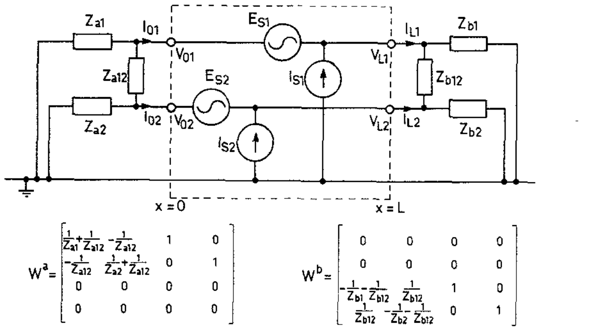

Fig. 2. Example of a multiconductor transmission line ($n = 2$) with lumped coupling at the boundaries.

We note also that the foregoing procedure, with particular reference to (4), (7), and (9) applies to nonuniform multiconductor lines as well; while the numerical determination of a suitable fundamental matrix is not straightforward [7], the technique is systematic.

III. APPLICATIONS

With the view of illustrating the procedure with the aid of examples which may be readily verified, we consider first coupled microstrip lines for which scattering parameters have been presented by Napoli [8] and Levy [9] using different procedures. The problem can be reduced to finding voltages and currents at $x = 0$ and $x = L$ of a single transmission line of length L, having either an odd mode characteristic impedance Z_{0o} or an even mode impedance Z_{0e}, a voltage source at one end and terminated in impedances Z_o at both ends. It can be very easily solved applying (10)–(15) when it is noted that each matrix reduces to a single number. Thus we find with the aid of (4a) and (9), letting $I_S = 0$ and assuming for generality that the terminations are Z_a and Z_b, respectively, rather than identical (while the characteristic impedance is Z_0), that

$$V(x) = \frac{E_S(0)Z_0 \{\exp(-\gamma x) + \rho_B \exp[\gamma(x - 2L)]\}}{(Z_a + Z_0)[1 - \rho_A\rho_B \exp(-2\gamma L)]} \quad (16)$$

$$I(x) = \frac{E_S(0) \{\exp(-\gamma x) - \rho_B \exp[\gamma(x - 2L)]\}}{(Z_a + Z_0)[1 - \rho_A\rho_B \exp(-2\gamma L)]} \quad (17)$$

where γ represents one of the two eigenvalues of the problem obtained by [10] solving $\det[\gamma^2 U - ZY] = 0$ and $\rho_A = (Z_a - Z_0)/(Z_a + Z_0)$, while $\rho_B = (Z_b - Z_0)/(Z_b + Z_0)$.

As a second example arising in connection with the Chipman's method [11],[12] of impedance measurement we consider a single transmission line of length L terminated in Z_a at $x = 0$ and Z_b at $x = L$; voltage is induced from a loop assumed to be located at a point ξ and it is desired to find the current at a point $x > \xi$.

Applying (10)–(15) and noting that each matrix reduces to a single number we find easily that

$$I(x) = \frac{E_S(\xi)}{2Z_0[1 - \rho_A\rho_B \exp(-2\Gamma L)]} \times \{\exp[\Gamma(\xi - x)] - \rho_A \cdot\exp[-\Gamma(x + \xi)] - \rho_B \exp[\Gamma(x + \xi - 2L)] + \rho_A\rho_B \cdot\exp[\Gamma(x - \xi - 2L)]\} \quad (18)$$

where ρ_A and ρ_B have the same significance as before; this result has been derived by Jackson [11] with some changes of notation.

Finally, it may be observed that when W^a and W^b are suitably chosen, letting $E_S = 0$ and $I_S = 0$ in turn, yields the open-circuit and short-circuit matrices of the system, respectively.

Thus letting $E_S = 0$ and noting that in this case

$$V(x) = \int_0^L G_{12}(x,\xi) I(\xi)\, d\xi \quad (19)$$

if $I(\xi)$ represents a source of unit magnitude (and is interpreted as n unit vectors times $\delta\xi$) located at $\xi = 0$ and $\xi = L$, respectively, then $V(x)$ is numerically equal to $G_{12}(x,\xi)$ and all open-circuit parameters are arrived at by a suitable choice of x and ξ. For example, for a single transmission line it is easily verified that

$$G_{12}(x,\xi) = \frac{Z^{1/2}Y^{-1/2} \cosh \Gamma x \cosh \Gamma(L - \xi)}{\sinh \Gamma L}, \qquad \xi \geq x \quad (20)$$

and hence $z_{11} = G_{12}(0,0)$, $z_{12} = G_{12}(0,L)$, while z_{21} and z_{22} follow from $G_{12}(x,\xi)\, \xi \leq x$ (or in this case by symmetry).

The short circuit parameters can be analogously derived with the aid of $G_{21}(x,\xi)$ when W^a and W^b are suitably chosen. However, it should be noted that both the open-circuit and short-circuit matrices may be also derived more directly with the aid of the transition matrix (5) without reference to (4).

IV. CONCLUSIONS

The Green's matrix procedure, when applied to multiconductor transmission lines, facilitates their analysis in a highly systematic and efficient manner. Thus we note, for example, that conceptually the derivation of the analog of (18) corresponding to more than a single transmission line presents no fresh difficulties, while the use of alternative techniques could prove cumbersome.

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Design and Analysis of a Waveguide-Sandwich Microwave Filter

YUSUKE TAJIMA AND YOSHIHIKO SAWAYAMA

***Abstract*—In 1972 Konishi proposed a unique waveguide filter of sandwich-like construction that was attractive because of its simplicity. This short paper presents an analysis of the sandwich filter that has a conductive sheet with finite thickness sandwiched between waveguide shells. An equivalent circuit is derived, design charts are proposed, and Young's technique is applied to the design of an *M*-band waveguide filter. The experimental results were found to be in good agreement with the theory and analysis techniques developed herein.**

I. INTRODUCTION

Microwave bandpass filter design techniques have been developed by many authors. For instance, Cohn [1] and Riblet [2] have treated coupled-resonator-microwave filters with narrow and moderate bandwidths, while Young [3] has developed a more general technique that holds for both small and wide bandpass microwave filters. However, the structures of coupled resonator filters are usually fairly complicated because many parts are required and the dimensions of each are critical to the filter's performance.

In 1972 Konishi proposed [4] the "microwave filter with mounted planar circuit in a waveguide," and similar structures have also been suggested bv Meier [5]. These circuits are essentially com-

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